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STABILITY OF COMPACT LEAVES WITH TRIVIAL LINEAR HOLONOMY

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SEVERAL THEOREMS stating that under certain hypotheses a compact leaf L of a foliation is stable under small perturbations are extended by replacing hypotheses on the fundamental group of L by conditions on the first real homology $H_1(L;\mathbb{R})$, provided that L has trivial linear holonomy and the perturbation is C^1 small. The compact leaf may be the fiber of a fibration or an isolated compact leaf. The proofs use the basic construction of Thurston's Generalized Reeb Stability Theorem.

§0. INTRODUCTION

Let F_0 be a C^1 codimension q foliation of a smooth manifold M and let L_0 be a compact leaf of F_0. We say that L_0 is a *stable compact leaf* if every foliation F sufficiently C^1 close to F_0 has a compact leaf near L_0. The Thurston–Langevin–Rosenberg generalization of the Reeb Stability Theorem asserts that if the linear holonomy of L_0 is trivial and $H^1(L_0, \mathbb{R})$ vanishes then L_0 is a stable compact leaf [15, 31]. We use the essential idea of this theorem, as simplified by Reeb and others [24, 25], to obtain a further generalization (Theorem 1, which we call the Conditional Stability Theorem) and various corollaries to the effect that certain compact leaves are stable.

Recall that if g belongs to the set $\Omega(L_0, x_0)$ of loops on L_0 based at a point $x_0 \in L_0$, then the *holonomy* $H(g)$: $U \to D$ of g is a local diffeomorphism, defined on an appropriate neighborhood U of x_0 in a q-dimensional disk D in M transverse to F_0 and centered at x_0, obtained by lifting g to nearby leaves of F_0. Similarly, given another foliation F sufficiently close to F_0, the *perturbed holonomy* $H_F(g)$: $U \to D$ is defined by lifting g to leaves of F (see §1 for fuller details). Throughout this paper we shall suppose that L_0 has *trivial linear holonomy*, i.e. for each loop $g \in \Omega(L_0, x_0)$ the derivative of the holonomy $DH(g)_{x_0}$ is the identity automorphism on the tangent plane TD_{x_0} to D at x_0. Let $\mathrm{Fol}_q^1(M)$ be the space of C^1 codimension q foliations of M with the C^1 topology (see §1). We can now state the basic lemma from which all the other theorems of this paper follow.

THEOREM 1. *The Conditional Stability Theorem. Let $g_1, \ldots, g_k \in \Omega(L_0, x_0)$ be loops whose real homology classes $\bar{g}_1, \ldots \bar{g}_k$ generate $H_1(L_0;\mathbb{R})$. Suppose that L_0 has trivial linear holonomy. Then for any neighborhood N of L_0 in M there exist neighborhoods V of F_0 in*

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$Fol_q^1(M)$ and U of x_0 in D, such that for all $F \in V$, we have:

1. *The perturbed holonomy $H_F(g_i)$: $U \to D$ is defined for $i = 1, \ldots k$;*
2. *If for some $x \in U$, $H_F(g_i)(x) = x$ for all i $(1 \le i \le k)$, then the leaf L of F passing through x is contained in N, diffeomorphic to L_0, and hence compact.*

When $H_1(L_0: \mathbb{R})$ vanishes the hypothesis $H_F(g_i)(x) = x$ becomes vacuous ($k = 0$), and we recover the theorem of Thurston–Langevin–Rosenberg, whose proof inspired the proof of the Conditional Stability Theorem.

THEOREM 2. (*Langevin and Rosenberg* [15]). *If a compact leaf L_0 of F_0 has trivial linear holonomy and satisfies $H_1(L_0; \mathbb{R}) = 0$, then there exist neighborhoods V of F_0 in $Fol_q^1(M)$ and U of x_0 in D such that for every $F \in V$, the union of the leaves of F meeting U is an open neighborhood of L_0 foliated as a product by F with all the leaves diffeomorphic to L_0.*

In particular, L_0 is a stable compact leaf.

Thurston's conclusion [31] that L_0 has a neighborhood foliated as a product by F_0 is the special case in which F is F_0.

In Theorem 2, L_0 is *intrinsically stable*, i.e. its stability follows from its topology alone (along with the hypothesis that the linear holonomy is trivial). Often the stability of a compact leaf depends on the surrounding foliation, either *locally* or *in the large*. For example, Seifert's Theorem [29] asserts that any foliation near to the Hopf fibration of the three-dimensional sphere S^3 has a compact leaf near to one of the Clifford circle fibers. In this spirit, we shall say that the fiber L of a smooth fibration $p: M \to B$ of compact manifolds is *stable in the large* if every foliation F of M sufficiently C^1 close to the foliation F_0 of M by the fibers of p has a compact leaf near to one of the original fibers.

The problem of when such stability in the large holds was posed by Rosenberg (Problem 11, p. 244 of [16]). Various affirmative results have been obtained with hypotheses on the base and on the fiber or its fundamental group [2, 3, 8, 16, 22]. In §3 we shall prove two versions of the Conditional Stability Theorem for fibrations (Theorems 3 and 3′) and in later sections use them to extend some of these results. In this context we shall always suppose that $p: M \to B$ is a smooth fibration of connected compact manifolds with fiber L and basepoint $x_0 \in L$ such that B has empty boundary. Theorems 4, 5 and 6 (the last two cited without proof from Suely Druck's thesis [4, 5] and Bonatti and Haefliger [1]) are partial results in the direction of the following generalization of Seifert's Theorem.

CONJECTURE 0.1. If $p: M \to B$ is a fibration as above such that $H_1(L; \mathbb{R}) \approx \mathbb{R}$ and the Euler characteristic $\chi(B)$ is nonzero, then L is stable in the large.

In order to state Theorem 4, we shall assume the following hypothesis, which permits the choice of a suitable element in the fundamental group $\pi_1(L_x, x)$ of each fiber. Let $y_0 = p(x_0)$ be the basepoint of B.

HYPOTHESIS 0.2. $H_1(L; \mathbb{R}) \approx \mathbb{R}$ and the natural action of $\pi_1(B, y_0)$ on the center Z of $\pi_1(L, x_0)$ fixes an element $\tilde{g} \in Z$ whose real homology class $\bar{g}$ generates $H_1(L; \mathbb{R})$.

THEOREM 4. *Assume Hypothesis 0.2 and suppose that $\chi(B) \neq 0$. If the dimension of B is 2 or p admits a global section, then L is stable in the large.*

This theorem was inspired by Langevin and Rosenberg [16] who proved stability in the large of the fiber L under C^0 perturbations in the special case when $\pi_1(L, x_0) \approx \mathbb{Z}$ provided that B is a closed surface with $\chi(B)\neq 0$. Fuller [8] obtained the same conclusion when the base B is a closed manifold of arbitrary dimension with $\chi(B)\neq 0$ and when the fiber L is the circle.

The next theorem was proved by S. Druck using the Conditional Stability Theorem. On the other hand, Bonatti and Haefliger obtained Theorem 6 from Fuller's theorem by an analysis of the holonomy groupoid.

THEOREM 5. (*S. Druck* [4, 5]). *Assume* 0.2 *and suppose that L and M are oriented with empty boundary. Let $[L]\in H_k(L;\mathbb{Z})$ be the fundamental class of the fiber and i: $L\to M$ the inclusion map. If $0\neq\chi(B)\cdot i_*[L]\in H_k(M;\mathbb{Z})$, then L is stable in the large.*

The hypothesis that M is oriented can be dropped by passing to a double cover of M if $2\chi(B)\cdot i_*[L]\neq 0$.

THEOREM 6. (*Bonatti and Haefliger* [1]). *Suppose that the commutator group $[\pi,\pi]$ of the fundamental group $\pi=\pi_1(L,x_0)$ of the fiber is finitely generated and that $Hom([\pi,\pi],\mathbb{R})=0$.* If $\mathrm{H}_1(L;\mathbb{R})\approx\mathbb{R}$ and $\chi(B)\neq 0$, *then L is stable in the large.*

It is not difficult to construct examples showing that Theorems 4, 5, and 6 are independent of each other (see 4.2).

In the codimension one case, every smooth fibration has the form $p{:}M\to S^1$ where M is the suspension of a diffeomorphism $f{:}L\to L$ of the fiber, $M=(L\times I)/\sim$, where $(x,1)\sim(f(x),0)$, and p is the obvious projection to the circle $S^1=\mathbb{R}/\mathbb{Z}$. In analogy with Kneser's Theorem [14] that every foliation of the Klein bottle has a compact leaf (i.e. a circle) we obtain the following.

THEOREM 7. *If $H_1(L;\mathbb{R})\approx\mathbb{R}$ and f_*: $H_1(L;\mathbb{R})\to H_1(L;\mathbb{R})$ is multiplication by a negative real number c, then L is stable in the large.*

(In fact, $c=-1$.) This theorem was inspired by Langevin and Rosenberg [16] who proved the analogous result for C^0 perturbations when $\pi_1(L,x_0)\approx\mathbb{Z}$. Plante [22] has proved a similar theorem, without restriction on the dimension of $H_1(L;\mathbb{R})$, provided $\pi_1(L,x_0)$ belongs to a certain class $\mathcal{S}$ of finitely generated groups, including polycyclic groups and finitely generated groups of non-exponential growth.

THEOREM 8. (*Plante* [22]). *If $\pi_1(L,x_0)\in\mathcal{S}$ and f_*: $H_1(L;\mathbb{R})\to H_1(L;\mathbb{R})$ has no positive real eigenvalues, then L is stable in the large under C^0 small perturbations.*

Theorem 7 includes fibrations not treated by Plante's theorem (see Example 5.2). It would be interesting to find a common generalization, but, as examples given in [22] show, some restriction on $\pi_1(L_1,x_0)$ is necessary if the fiber has first Betti number greater than one.

In the case of a fibration whose fiber L has commutative fundamental group, N. Desolneux–Moulis [2, 3] has obtained stability in the large of the fiber under a certain transversality condition. We obtain a generalization of her result (see Theorem 9, §6.)

We shall say that a compact leaf L_0 of a C^1 foliation F_0 of M is *locally stable* if L_0 is a stable compact leaf of F_0 restricted to every neighborhood N of L_0 foliated by the restriction of F_0, i.e. every foliation of N sufficiently C^1 close to F_0 restricted to N has a compact leaf diffeomorphic to L_0. For example, if a periodic orbit L_0 of a flow has the point $x_0\in L_0$ as a stable isolated fixed point of the Poincaré map (i.e. the holonomy) on a normal disk D through x_0, then L_0 is locally stable. Hirsch [13] has given a number of interesting theorems on local stability of a compact leaf L_0 in terms of properties of $\pi_1(L_0,x_0)$ and the holonomy. In a similar vein we obtain the following results.

THEOREM 10. *Let $\tilde{g} \in \pi_1(L_0, x_0)$ be such that its image $h(\tilde{g})$ under the Hurewicz homomorphism generates $H_1(L_0; \mathbb{R}) \approx \mathbb{R}$. Suppose that x_0 is a stable isolated fixed point of the holonomy map $H(\tilde{g})$ and that the compact leaf L_0 has trivial linear holonomy. Then L_0 is locally stable.*

Recall that an element g of a group G is accessible [13] in G if there exists a finite sequence of subgroups $G_0 \lhd G_1 \lhd \ldots \lhd G_r \subseteq G$ such that g generates G_0, G_{i-1} is normal in $G_i (i = 1, \ldots r)$ and G_r has finite index in G. Let $\langle g_1, \ldots, g_k \rangle$ denote the subgroup generated by elements $g_1, \ldots, g_k \in G$.

THEOREM 11. *Suppose that $\tilde{g}_1, \ldots, \tilde{g}_k$ are elements of $\pi_1(L_0, x_0)$ such that their images $h(\tilde{g}_1), \ldots, h(\tilde{g}_k)$ generate $H_1(L_0; R)$ and such that $\tilde{g}_1$ is accessible in $\langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle$. If x_0 is a stable isolated fixed point of $H(\tilde{g}_1)$, the codimension of F_0 is one, and the compact leaf L_0 has trivial linear holonomy, then L_0 is a locally stable compact leaf.*

COROLLARY. *Theorem 11 holds if the hypothesis that $\tilde{g}_1$ is accessible is replaced by the hypothesis that $\tilde{g}_1$ commutes with $\tilde{g}_2, \ldots, \tilde{g}_k$.*

Theorem 10, which extends Theorem 1.2(d) of [13], follows as an immediate corollary from Theorem 1. An isolated fixed point is stable, of course, iff its fixed point index is non-zero. Theorem 11, which extends part of Theorem 1.4 of [13], and Theorem 12, which treats local stability of a compact leaf in a one parameter family $F(t)$ of foliations under a certain genericity condition at $t = 0$, are proven in §7.

Up to now we have only considered stability of compact leaves under perturbations that are small (in the C^1 or C^0 topology). Certain closed 3-manifolds and one 2-manifold (the Klein bottle) have the property that *every* codimension one foliation has a compact leaf [14,17,18,19,32]. On the other hand, on manifolds of dimension greater than three every codimension one foliation is homotopic to a C^0 foliation with no compact leaves (and with all leaves smooth) [26,27]. Every foliation of codimension greater than one (excluding the trivial foliation by points) is homotopic to a foliation without compact leaves [28], which can be taken to be C^∞ if the codimension is greater than two, and C^2 [12, 11] if the codimension is exactly two.

We have only considered the case of trivial linear holonomy but it is not difficult to apply the Conditional Stability Theorem to a compact leaf L with finite linear holonomy by passing to a finite covering space $\tilde{L}$ that trivializes the linear holonomy, but then we must use enough loops to generate $H_1(\tilde{L}; \mathbb{R})$. It is also easy to adapt Theorems 3 and 3′ to treat Hausdorff compact foliations (foliations with every leaf compact with finite holonomy, see [6]).

D. Stowe [30] has given a remarkable extension of Thurston's Generalized Reeb Stability Theorem, by showing that a compact leaf L is C^1 stable if $H^1(L, \mathbb{R}^q_\gamma) = 0$ where the coefficients are twisted by the linear holonomy representation γ: $\pi_1(L, x_0) \to \mathrm{Aut}(\mathbb{R}^q)$,

$$\gamma(\tilde{g}) = DH(\tilde{g})_{x_0}: TD_{x_0} \to TD_{x_0},$$

where we identify TD_{x_0} with $\mathbb{R}^q$. It would be interesting to combine Stowe's theorem with the idea of conditional stability.

The structure of the remainder of the paper is as follows. Definitions of holonomy and perturbed holonomy are given in §1 and used to prove Theorem 1 in §2. Theorems 3 and 3′, which extend conditional stability to stability in the large of the fiber of a fibration, are stated and proved in §3. Their corollaries, Theorems 4, 7 and 9, are proven in §4, §5 and §6, respectively, where several examples are also given. Theorems 11 and 12, which treat local stability of compact leaves, are proven in §7, while some lemmas are proven in the Appendix.

§1. HOLONOMY AND PERTURBED HOLONOMY

Throughout this paper, M will be a smooth (C^∞) manifold and F_0 a C^1 codimension q foliation of M with a compact leaf L_0 and basepoint x_0 in L_0. For convenience we shall fix a Riemannian metric on M, but its choice will not affect the results.

The normal bundle to F_0 in M is, in general, only a C^0 vector bundle, but it can easily be approximated by a nearby C^1 vector bundle ν. We may take ν to be a subbundle of the tangent bundle TM of M, and choose it to be transverse to TF_0, the tangent bundle to the leaves of F_0. For $\varepsilon>0$ let $D(\nu, L_0, \varepsilon)$ be the subspace of ν consisting of vectors over L_0 of length less than ε. Choose ε sufficiently small so that the restriction of the exponential map exp: $TM \to M$ takes $D(\nu, L_0, \varepsilon)$ diffeomorphically onto an open neighborhood N of L_0. Let $\pi: N \to L_0$ be the bundle projection induced by the tangent bundle projection and take ε sufficiently small so that the open q-disk fibers of π are transverse to F_0. Thus (N, π) is a tubular neighborhood of L_0. Let $D=\pi^{-1}(x_0)$ be the q-disk fiber passing through x_0.

We shall consider foliations near to F_0 in the space $Fol_q^1(M)$ of all C^1 codimension q foliations of M endowed with the Epstein compact C^1 topology which satisfies the axioms given in [7]. The first axiom (which we shall not need) asserts that the group of C^1 diffeomorphisms of M acts continuously on $Fol_q^1(M)$. The second axiom, which states roughly that every foliation F sufficiently C^1 close to F_0 has perturbed holonomy defined and C^1 close to that of F_0, is closely related to the following lemma, whose proof is given in the Appendix. This lemma is basic to the concept of perturbed holonomy, and implicit in virtually all results on stability of compact leaves. We say that a path $g: I \to L_0$ *lifts along* π to a path $g': I \to N$, or equivalently that g' is a *lift* of g, if $\pi \circ g' = g$. We shall suppose that all paths and loops are *piecewise smooth* with domain $I=[0,1]$ throughout this paper.

LEMMA 1.1. *Given $C'>0$ there exist neighborhoods V of F_0 in $Fol_q^1(M)$ and U of x_0 in D with the following property. For every $F \in V$ and $x \in U$, every path g in L_0 of length less than C' starting at x_0 lifts along π to a path g' starting at x and lying on a leaf of F.*

We may now define holonomy and perturbed holonomy. Let g be a path on L_0 from x_0 to y_0 of length less than C', and let V and U be the neighborhoods given by the lemma.

Definition 1.2. The perturbed holonomy of the path g with respect to the foliation F is the map $H_F(g)$: $U \to \pi^{-1}(y_0)$ defined by setting $H_F(g)(x) = g'(1)$ where g' is the lift of g starting at x and lying on the leaf of F which passes through x. If we set $F=F_0$ then we obtain the *holonomy* $H(g) = H_{F_0}(g)$ *of* g.

It is easy to see that g' is uniquely determined, since the point $g'(t)$ must lie both on the leaf of F through x and on the transverse disk $\pi^{-1}[g(t)]$. The following properties of the perturbed holonomy are clear consequences of the definition, and of course hold for the holonomy as a special case. Let D_x denote the transverse disk $\pi^{-1}(x)$ through $x \in L_0$.

PROPOSITION 1.3. *Let g and g' be paths on L_0 with $g(0)=x_0$, and let F be sufficiently C^1 close to F_0. Then the following properties hold.*

1. *$H_F(g): U \to D_x$ is a C^1 diffeomorphism onto an open subset of D_x, where $x=g(1)$.*
2. *If $g(1)=g'(0)$ then $H_F(g') \circ H_F(g) = H_F(gg')$ wherever the right hand side is defined.*
3. *If e is the constant path at x_0, then $H_F(e)$ is the identity map.*
4. *If g^{-1} is the inverse path of g, then $H_F(g^{-1}) = H_F(g)^{-1}$.*
5. *If $g_t (0 \leqslant t \leqslant 1)$ is a homotopy of paths from x_0 to y_0 in L_0 and x is a point in D such that each g_t lifts along π to a path g'_t starting at x and lying on the leaf of F through x, then $H_F(g_0)(x) = H_F(g_1)(x)$.*

The following property of the perturbed holonomy, which is related to Epstein's second axiom for the C^1 topology on the space of foliations, will be proven in the Appendix.

PROPOSITION 1.4. *Let g be a path on L_0 from x_0 to x and let V and U be the neighborhoods given by Lemma 1.1 where C' is chosen to be greater than the length of g. Then the perturbed holonomy defines a continuous map $H(g)$: $V \to C^1(U, D_x)$, $H(g)(F) = H_F(g)$: $U \to D_x$, where $C^1(U, D_x)$ is the space of C^1 maps $U \to D_x$ with the C^1 topology.*

We also define the *linear holonomy* $DH(g)_x$: $TD_x \to TD_y$ of a path g on L_0 from x to y to be the derivative of the holonomy map $H(g)$ at the point x. In particular, if g belongs to the space $\Omega(L_0, x_0)$ of loops on L_0 based at x_0, then $DH(g)_x$: $TD_x \to TD_x$. Throughout this paper we shall assume that L_0 has *trivial linear holonomy*, that is, for every loop $g \in \Omega(L_0, x_0)$ the linear holonomy $DH(g)_{x_0}$ is the identity automorphism of the tangent plane TD_{x_0}.

§2. PROOF OF THE CONDITIONAL STABILITY THEOREM

Before beginning the proof we state a fundamental lemma which is the basis of virtually all compact leaf stability theorems [29, 13, 31, 15 . . .]. Its statement is quite similar to the statement of the Conditional Stability Theorem, but the real homology of L_0 is replaced by the fundamental group and there is no hypothesis about the linear holonomy. As always, L_0 is a compact leaf of F_0 with basepoint x_0. Let $\tilde{g} \in \pi_1(L_0, x_0)$ denote the based homotopy class of a loop $g \in \Omega(L_0, x_0)$.

THE COMPACT LEAF STABILITY LEMMA 2.1. *Let $g_1, \ldots, g_m \in \Omega(L_0, x_0)$ be loops whose homotopy classes $\tilde{g}_1, \ldots, \tilde{g}_m$ generate $\pi_1(L_0, x_0)$. Then for any tubular neighborhood (N, π) of L_0 there exist neighborhoods V_1 of F_0 in $Fol^1_q(M)$ and U_1 of x_0 in D, such that for all $F \in V_1$ and $x \in U_1$, we have:*

1′. *The perturbed holonomy $H_F(g_j)(x) \in D$ is defined for $j = 1, \ldots, m$;*
2′. *If $H_F(g_j)(x) = x$ for all $j = 1, \ldots, m$, then the leaf L of F passing through x is contained in N, diffeomorphic to L_0 under the projection π, and hence compact.*

This lemma is implicit in the argument of [15], p. 108, where a brief outline of its proof is given. For the sake of completeness we provide a somewhat more detailed proof in the Appendix.

To begin the proof of the Conditional Stability Theorem, suppose that $g_1, \ldots, g_k \in \Omega(L_0, x_0)$ are loops whose real homology classes $\bar{g}_1, \ldots, \bar{g}_k$ generate $H_1(L_0; \mathbb{R})$, where L_0 is a compact leaf of the C^1 codimension q foliation F_0 of M with basepoint x_0, and suppose that the linear holonomy $DH(g_i)_{x_0}$ is the identity automorphism of TD_{x_0} for $i = 1, \ldots, k$, where $D = \pi^{-1}(x_0)$ is the q-disk fiber of the tubular neighborhood N of L_0 under the fiber map π: $N \to L_0$. These are the hypotheses of Theorem 1, as stated in the Introduction.

Without loss of generality we suppose that the set of loops $\{g_1, \ldots, g_k\}$ is closed under inversion, by adding in the inverse loops if necessary. Choose loops $g_{k+1}, \ldots, g_m \in \Omega(L_0, x_0)$ such that the set $\{\tilde{g}_1, \ldots, \tilde{g}_m\}$ generates $\pi_1(L_0, x_0)$ and is invariant under inversion. We then have $\bar{g}_j = h(\tilde{g}_j)$ for $j = 1, \ldots, m$, where h: $\pi_1(L_0, x_0) \to (H_1(L_0; \mathbb{R})$ is the Hurewicz homomorphism for real homology, and $\bar{g}_j$ and $\tilde{g}_j$ denote the real homology and homotopy classes of the loop g_j.

Now let V_1 and U_1 be the neighborhoods of F_0 and x_0 given by Lemma 2.1. According to (1′), the perturbed holonomy $H_F(g_j)(x) \in D$ is defined whenever $F \in V_1$, $x \in U_1$, and $1 \leq j \leq m$, so conclusion (1) of the theorem holds for arbitrary subneighborhoods $U \subseteq U_1$ of x_0 and $V \subseteq V_1$ of F_0.

In order to prove the theorem by contradiction, let us suppose that the conclusion (2) fails to hold. Since $\mathrm{Fol}_q^1(M)$ has countable neighborhood bases, this means that we can find a sequence of foliations F_n in V_1 converging to F_0 and a sequence of points x_n in U_1 converging to x_0, such that

$$H_F(g_i)(x_n)=x_n, \quad i=1,\ldots,k; \quad n=1,2\ldots, \tag{2.2}$$

but the conclusion of (2) fails to hold for any F_n and x_n. In particular, the conclusion of (2′) of the lemma cannot hold, and so the hypothesis of (2′) must fail. Thus for each n there exists some j, necessarily greater than k, such that

$$H_F(g_j)(x_n)\neq x_n. \tag{2.3}$$

We now use the basic idea of Thurston [31] as reformulated in [24, 25] to construct a non-trivial homomorphism $\tilde{\varphi}\colon \pi_1(L_0,x_0)\to\mathbb{R}^q$.

LEMMA 2.4. *The conditions* (2.2) *and* (2.3) *imply that there exists a non-zero homomorphism* $\tilde{\varphi}\colon \pi_1(L_0,x_0)\to\mathbb{R}^q$ *such that* $\tilde{\varphi}(\tilde{g}_i)=0$ *for* $i=1,\ldots,k$.

Assuming this lemma we can now complete the proof of the Theorem. Since $\mathbb{R}^q$ is an $\mathbb{R}$-module, such a homomorphism $\tilde{\varphi}$ must factor through $H_1(L_0;\mathbb{R})\approx H_1(L_0;\mathbb{Z})\otimes\mathbb{R}$, that is, there exists a homomorphism $\bar{\varphi}\colon H_1(L_0;\mathbb{R})\to\mathbb{R}^q$ such that $\tilde{\varphi}=\bar{\varphi}\circ h$. The lemma asserts that $\tilde{\varphi}(\tilde{g}_i)$ vanishes for $i\leq k$ and the elements $\bar{g}_1,\ldots,\bar{g}_k$ generate $H_1(L_0;\mathbb{R})$ by hypothesis. Thus $\tilde{\varphi}$ is identically zero, contradicting Lemma 2.4, and the theorem follows. □

Proof of Lemma 2.4. Embed D as an open set in $\mathbb{R}^q$ by a C^1 diffeomorphism, and to simplify notation, identify D with its image in $\mathbb{R}^q$. Set

$$d_n=\max_{1\leq j\leq m}|H_n(g_j)\,(x_n)-x_n|$$

where we write H_n for the perturbed holonomy H_{F_n} and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^q$. By (2.3) d_n is non-zero so we may define

$$\varphi_n(g)=d_n^{-1}(H_n(g)\,(x_n)-x_n)\in\mathbb{R}^q$$

for every loop $g\in\Omega(L_0,x_0)$ for which the perturbed holonomy $H_n(g)\,(x_n)$ is defined. Of course $\varphi_n(g_j)$ is defined for all n and j, since $F_n\in V_1$ and $x_n\in U_1$.

The definitions guarantee that the points $\varphi_n(g_j)$ all lie in the unit disk in $\mathbb{R}^q$, so by repeatedly passing to subsequences we may assume that $\lim_{n\to\infty}\varphi_n(g_j)$ exists for $j=1,\ldots,m$. Now define

$$\varphi(g)=\lim_{n\to\infty}\varphi_n(g)\in\mathbb{R}^q$$

for those loops for which the limit exists. It will follow from the next lemma that this is the case for all loops $g\in\Omega(L_0,x_0)$.

LEMMA 2.5. *If* $g,g'\in\Omega(L_0,x_0)$ *are loops such that* $\varphi(g)$ *and* $\varphi(g')$ *are defined, then* $\varphi(gg')$ *is also defined and* $\varphi(gg')=\varphi(g)+\varphi(g')$.

Proof of Lemma 2.5. According to the definition of the derivative $DH_n(g')_{x_n}$ of $H_n(g')$ at x_n,

$$H_n(g')\,(y)=H_n(g')\,(x_n)+DH_n(g')_{x_n}\cdot(y-x_n)+o(|y-x_n|)$$

for $y \in \mathbb{R}^q$ near x_n. Setting $y = H_n(g)(x_n)$ and using $H_n(gg') = H_n(g') \circ H_n(g)$ we obtain

$$d_n^{-1}(H_n(gg')(x_n) - x_n) = d_n^{-1}(H_n(g')(x_n) - x_n) + DH_n(g')_{x_n} d_n^{-1}(H_n(g)(x_n) - x_n) + \eta$$

where $\eta = d_n^{-1} o(|H_n(g)(x_n) - x_n|)$. Observe that $\eta \to 0$ as $n \to \infty$ since by hypothesis the limit defining $\varphi(q)$ exists. Now $H_n(g')$ converges C^1 to $H(g')$, since F_n converges to F_0 in the C^1 topology, so $DH_n(g')_{x_n}$ converges C^0 to $DH(g')_{x_0}$, which is the identity automorphism since we are assuming that the linear holonomy of L_0 with respect to F_0 is trivial. As $n \to \infty$ the RHS converges to $\varphi(g') + \varphi(g)$, so the limit of the LHS exists and is clearly $\varphi(gg')$, completing the proof of Lemma 2.5. □

Returning to the proof of Lemma 2.4, we apply Lemma 2.5 repeatedly to conclude that $\varphi(p)$ is defined whenever the loop p is a monomial in the generators $g_1, \ldots, g_m$. Note that if two loops g^0, $g^1 \in \Omega(L_0, x_0)$ are homotopic and $\varphi(g^0)$ is defined, then $\varphi(g^1)$ is defined and $\varphi(g^0) = \varphi(g^1)$. This is true because the intermediate loops g^t $(0 \leqslant t \leqslant 1)$ of the homotopy will have uniformly bounded length, and so by Lemma 1.1 the perturbed holonomy $H_n(g^t)(x_n)$ will be defined for all t, and consequently independent of t, for all sufficiently large n. Since every loop $g \in \Omega(L_0, x_0)$ is homotopic to a monomial in the generators g_j, φ: $\Omega(L_0, x_0) \to \mathbb{R}^q$ is globally defined. Lemma 2.5 now shows that φ is a homomorphism. Since it is homotopy invariant, it passes to a quotient homomorphism $\tilde{\varphi}$: $\pi_1(L_0, x_0) \to \mathbb{R}^q$.

According to the definitions of d_n and φ_n, for each n there is a j such that $|\varphi_n(g_j)| = 1$. For some j, say j', this must occur for infinitely many n, so $|\tilde{\varphi}(\tilde{g}_{j'})| = 1$ and $\tilde{\varphi}$ is not zero. On the other hand, (2.2) implies that $\tilde{\varphi}(\tilde{g}_i) = \varphi(g_i) = 0$ for i $= 1, \ldots, k$, establishing Lemma 2.4. □

§3. PERTURBATION OF FOLIATIONS DEFINED BY FIBRATIONS

In this section we give two global forms of the Conditional Stability Theorem, namely Theorems 3 and 3′, which apply to fibrations with compact fiber.

Throughout this section we shall suppose that $p: M \to B$ is a C^∞ fibration with compact connected fiber L. Denote the fiber containing $x \in M$ by L_x, and let F_0 be the foliation of M whose leaves are the fibers. Fix a Riemannian metric in M once and for all. Let $\nu \subset TM$ be the normal bundle orthogonal to the fibers, and for each x in M let D_x be the image of the closed ε-disk in the fiber ν_x under the exponential map exp: $TM \to M$. (If M is not compact, then take ε to be a positive function on B. If the boundary of the fiber ∂F is nonempty, then choose the metric so that for each x in ∂M, D_x lies entirely in ∂M.) We choose $\varepsilon > 0$ to be sufficiently small so that each D_x is an embedded disk transverse to the foliation F_0 and so that $N_x = \cup \{D_y : y \in L_x\}$ is a tubular neighborhood of L_x fibered over L_x by the projection π_x: $N_x \to L_x$ with the disks $D_y (y \in L_x)$ as fibers.

The fundamental groups $\pi_1(L_x, x)$ of the fibers are the stalks of a locally trivial sheaf q: $\Pi \to M$ of groups over M. Suppose that M is compact and let $\tilde{g}$: $M \to \Pi$ be a global section of this sheaf. Then for every foliation F sufficiently C^1 close to F_0 there is a *global perturbed holonomy map* $\tilde{H}_F(\tilde{g})$: $M \to M$ defined as follows. For each x in M choose a piecewise smooth loop $g_x \in \Omega(L_x, x)$ representing $\tilde{g}(x) \in \pi_1(L_x, x)$ so that the lengths of the loops g_x are uniformly bounded. Let g_{xF} be the lift of g_x along π_x to the leaf L_{xF} of F passing through $x = g_{xF}(0)$. Then define $\tilde{H}_F(\tilde{g})(x) = g_{xF}(1) \in D_x$. As before, the real Hurewicz homomorphism h: $\pi_1(L_x, x) \to H_1(L_x; \mathbb{R})$ takes $\tilde{g}(x)$ to an element $\bar{g}(x) = h(\tilde{g}(x)) \in H_1(L_x; \mathbb{R})$. Throughout this paper we denote loops (or paths) by $g, g', g_i, \ldots$, the corresponding homotopy classes by $\tilde{g}, \tilde{g}', \tilde{g}_i, \ldots$, and their real homology classes by $\bar{g}, \bar{g}', \bar{g}_i, \ldots$. We can now enunciate

THEOREM 3. *The Conditional Stability Theorem for Fibrations. Let $\tilde{g}_1, \ldots, \tilde{g}_k$ be global sections of the sheaf Π such that for some (and hence all) x in M, the images $\bar{g}_1(x), \ldots, \bar{g}_k(x)$*

generate $H_1(L_x;\mathbb{R})$, and let M be compact. Then for any $\varepsilon>0$ there exists a neighborhood V of F_0 in $Fol_q^1(M)$ such that for all $F\in V$ and $x\in M$ we have:

1. *The perturbed holonomy $\tilde{H}_F(\tilde{g}_i)\in D_x$ is defined for $i=1,\ldots,k$;*
2. *If $\tilde{H}_F(\tilde{g}_i)(x)=x$ for all i, $1\leqslant i\leqslant k$, then the leaf L_{xF} of F passing through x is contained in the normal ε-neighborhood N_x of L_x, diffeomorphic to L_x under the projection π_x: $N_x\to L_x$, and hence compact.*

When there are not enough global sections of Π to generate $H_1(L_x;\mathbb{R})$, or when the total space M is not compact, the following variant of Theorem 3 may be applicable. Let K be a compact metric space and $f: K\to M$ a continuous map. By a lift of f to Π we mean a map $\tilde{g}$: $K\to\Pi$ such that $q\circ\tilde{g}=f$. For a foliation F of M sufficiently C^1 close to F_0 we may define the *perturbed holonomy map* $H'_F(\tilde{g})$: $K\to M$. For $z\in K$ we define $H'_F(\tilde{g})(z)\in D_x$, $x=f(z)$, by lifting a loop $g_z\in\Omega(L_x,x)$ representing $\tilde{g}(z)\in\pi_1(L_x,x)$ along π_x: $N_x\to L_x$ to the loop g_{zF} on the leaf L_{xF}, with $g_{zF}(0)=x$, and then setting $H'_F(\tilde{g})(z)=g_{zF}(1)$. We can now state

THEOREM 3′. *Let $\tilde{g}_1,\ldots,\tilde{g}_k$: $K\to\Pi$ be lifts of f to Π such that for each z in K the images $\tilde{g}_1(z),\ldots,\tilde{g}_k(z)$ generate $H_1(L_x;\mathbb{R})$, $x=f(z)$. Then for any $\varepsilon>0$ there exists a neighborhood V of F_0 in $Fol_q^1(M)$ such that for all $F\in V$ and $z\in K$, setting $x=f(z)$, we have:*

1. *The perturbed holonomy $H'_F(\tilde{g}_i)(z)\in D_x$ is defined for $i=1,\ldots,k$;*
2. *If $H'_F(\tilde{g}_i)(z)=x$ for all i, $1\leqslant i\leqslant k$, then the leaf L_{xF} of F passing through x is contained in N_x, diffeomorphic to L_x under the projection π_x: $N_x\to L_x$, and hence compact.*

Since Theorem 3 is identical to the special case of Theorem 3′ in which $K=M$ and $f=Id$: $K\to M$, it suffices to prove Theorem 3′. Before beginning the proof we shall state a proposition which implies that the perturbed holonomy map $H'_F(\tilde{g}_i)$: $K\to M$ is well-defined and continuous.

The *fundamental groupoid* Π_1 of M is the set of all homotopy classes (with stationary endpoints) of paths in M, with the smooth structure of a locally trivial sheaf over $M\times M$ given by the projection (a_0,a_1):$\Pi_1\to M\times M$ defined $a_i(\tilde{g})=g(i)$ $(i=0,1)$, where g: $I\to M$ is a path representing $\tilde{g}\in\Pi_1$. The product $\tilde{g}'\tilde{g}''$ of two elements of Π_1 is defined, using path multiplication, provided $a_1(\tilde{g}')=a_0(\tilde{g}'')$.

Suppose that K is a compact metric space and $\tilde{g}$:$K\to\Pi_1$ a continuous map. Recall that D_x is the transverse disk of radius ε centered at x, and for $\delta>0$ let $D_x(\delta)$ be the closed concentric subdisk of radius δ. Define

$$A_i(\delta)=\{(z,x')\in K\times M\colon x'\in D_x(\delta),\text{ where } x=a_i(\tilde{g}(z))\}$$

for $i=0,1$. For $(z,x')\in A_0(\delta)$, $0<\delta<\varepsilon$, and for F sufficiently C^1 close to F_0, define the perturbed holonomy $H_F(\tilde{g})(z,x')\in A_1(\varepsilon)$ to be the endpoint of the path starting at x' obtained by lifting a path g_z representing $\tilde{g}(z)$ to the leaf of F through x'.

PROPOSITION 3.1. A. *Given $\tilde{g}$: $K\to\Pi_1$ as above and $0<\delta<\varepsilon$, there exists a neighborhood V of F_0 in $Fol_q^1(M)$ such that for all $F\in V$ the perturbed holonomy map $H_F(\tilde{g})$: $A_0(\delta)\to A_1(\varepsilon)$ is well-defined and continuous.*

B. *If in addition K is a smooth manifold and $\tilde{g}$: $K\to\Pi_1$ is C^1, then $H_F(\tilde{g})$ is C^1 and the induced map*

$$H\colon V\to C^1(A_0(\delta),\,A_1(\varepsilon))$$

is continuous, where $C^1(A_0(\delta),\,A_1(\varepsilon))$ is the space of C^1 maps with the C^1 topology.

The proof of this Proposition is given in the Appendix.

Proof of Theorem 3′. Note that Π is contained in Π_1, as the subspace of homotopy classes of paths that are loops. Hence the hypotheses of Theorem 3′ yield maps $\tilde{g}_i$: $K \to \Pi_1$, $i = 1, \ldots, k$. The perturbed holonomy map $H'_F(\tilde{g}_i)$: $K \to M$ of the Theorem is obtained as the composition of the canonical map $K \to \mathrm{Graph}\,(f) \subset A_0(\delta)$, followed by $H_F(\tilde{g}_i)$: $A_0(\delta) \to A_1(\varepsilon)$ of the proposition, and then by the projection on the second factor p_2: $A_i(\varepsilon) \subset K \times M \to M$. Consequently the proposition implies that there exists a neighborhood V of F_0 such that for all F in V, $H'_F(\tilde{g}_i)$: $K \to M$ is well-defined and continuous for $i = 1, \ldots, k$.

Now suppose that Theorem 3′ is false, so that (2) fails to hold, in order to arrive at a contradiction. Then there exist sequences F_n of foliations in V converging in $\mathrm{Fol}^1_q(M)$ to F_0 and points z_n in K, such that $H'_{F_n}(\tilde{g}_i)(z_n) = x_n$ for all i, where $x_n = f(z_n)$, but the leaf L_n of F_n passing through x_n fails to be contained in N_{x_n} or is not diffeomorphic to L_{x_n} under the projection π_{x_n}: $N_{x_n} \to L_{x_n}$. By passing to a subsequence we may suppose that the sequence z_n converges to a point z in K. Now $H'_{F_n}(\tilde{g}_i)(z_n) = x_n$ implies that $H_{F_n}(g_{iz})(x_n) = x_n$, where $H_{F_n}(g_{iz})$ is the perturbed holonomy defined in §1 by lifting the loops g_{iz} to nearby leaves of F_n. By hypothesis the homology classes $\bar{g}_{iz} = \bar{g}_i(z)$ $(i = 1, \ldots, k)$ generate $H_1(L_x; \mathbb{R})$, $x = f(z)$, so Theorem 1 asserts that for sufficiently large n, L_n is contained in N_x and diffeomorphic to L_x under π_x. This clearly shows that for sufficiently large n, L_n is contained in N_{x_n} and diffeomorphic to L_{x_n} under the projection π_{x_n}. This contradiction establishes Theorem 3′. □.

Note that K is not assumed to be connected, and in applying Theorem 3′ it is possible to choose different sets of generators $\{\tilde{g}_i\}$ of the fundamental group of the fiber on distinct components of K (at least if K is locally connected). We shall do this in proving Theorem 9.

§4. PROOF OF THEOREM 4

In this section we prove Theorem 4 and give several examples.

Proof of Theorem 4. Since the element $\tilde{g}$ mentioned in (0.2) belongs to the center Z of $\pi_1(L, x_0)$ and is invariant under the action of $\pi_1(B, y_0)$, there is a global section $\tilde{g}_1$ of the sheaf Π of fundamental groups of the fibers over M such that $\tilde{g}_1(x_0) = \tilde{g}$. By hypothesis the corresponding real homology class $\bar{g}_1(x_0) = \bar{g}$ generates $H_1(L, \mathbb{R})$. According to Theorem 3 with $k = 1$, there exists a neighborhood V of F_0 such that for any $F \in V$, the global perturbed holonomy $\tilde{H}_F(\tilde{g}_1)$: $M \to M$ is defined and any fixed point x of $\tilde{H}_F(\tilde{g}_1)$ lies on a compact leaf L_{xF} of F inside the tubular neighborhood N_x of L_x and diffeomorphic to L_x. To complete the proof of the theorem it suffices to find such a fixed point.

In the second case of the theorem, when there is a global section s: $B \to M$, the map $p \circ \tilde{H}_F(\tilde{g}_1) \circ s$: $B \to B$ must have a fixed point $y \in B$, because $\chi(B) \neq 0$. Then $x = s(y)$ will be a fixed point of $\tilde{H}_F(\tilde{g}_1)$.

In order to prove the first case, when the base B has dimension 2, recall that $\tilde{H}_F(\tilde{g}_1)(x)$ lies on the tranverse disk D_x which was defined by the exponential map $\exp_x$ on $v_x = T(L_x)^{\perp}_x \subset TM_x$. Consequently there is a well-defined vector field X on M, orthogonal to the fibers, such that $\tilde{H}_F(\tilde{g}_1)(x) = \exp_x X(x)$ for each point x in M, and it suffices to find a point where X vanishes.

Let us suppose that X is nonvanishing on M, in order to arrive at a contradiction. Define a homomorphism I_F: $\pi_1(L, x_0) \to \mathbb{Z}$ by letting $I_F(\tilde{c})$ be the number of times that $p_* X(c(t))$ winds around the origin of TB_{y_0} as t goes from 0 to 1, where c: $I \to L$ is a loop representing $\tilde{c} \in \pi_1(L, x_0)$, and we have fixed an orientation of TB_{y_0}. We shall use Seifert's argument [29] (see also [16], Lemma 2.1) to show that I_F vanishes.

Let D' be a closed disk in B containing y_0 in its interior, sufficiently small so that $p^{-1}(D')$ is contained in the normal neighborhood $N = N_{x_0}$ around L. The two projections $\pi = \pi_{x_0}$: $N \to L$

and $p\colon M \to B$ define a diffeomorphism $h=(\pi, p)\colon p^{-1}(D') \to L \times D'$. Let $g\colon S^1 \to L$ be a smoothly embedded loop representing $\tilde{g} \in \pi_1(L, x_0)$. Then $f = h^{-1} \circ (g \times Id)\colon S^1 \times D' \to p^{-1}D' \subset N$ is a smooth embedding transverse to F. Thus F induces a foliation F' by curves transverse to the disks $f(\{x\} \times D')$ on the solid torus $T' = f(S^1 \times D')$. For some small disk neighborhood D'' of y_0, and possibly restricting F to lie in a smaller neighborhood V_1 of F_0, there will be a first return map $\pi''\colon T'' \to T'$ defined on the smaller solid torus $T'' = f(S^1 \times D'')$ by letting $\pi''(x')$ be the first point where the leaf of F' passing through x' in the direction of the loop g returns to the disk $f(\{x\} \times D'$ that contains x'.

On T'' the map π'' differs from $\tilde{H}_F(\tilde{g}_1)$ only by a small displacement along the leaves of F, so π'' has no fixed points and the Seifert index of the loop g, $I_{F'}(g)$, defined similarly to I_F, agrees with $I_F(\tilde{g})$. By an ingenious argument Seifert [29] shows that $I_{F'}(g)=0$, and from this we conclude that $I_F(\tilde{g})=0$. Now $I_F\colon \pi_1(L, x_0) \to \mathbb{Z}$ factors through $H_1(L; \mathbb{Z})$, which is a finitely generated abelian group of rank one (since $H_1(L; \mathbb{R}) \approx \mathbb{R}$), and the image of $\tilde{g}$ in $H_1(L; \mathbb{Z})$ has infinite order, so I_F must be the zero homomorphism.

We now show that I_F cannot vanish identically, to obtain the desired contradiction. There is a partial section $s\colon B' \to M$ of p over $B' = B - \mathrm{Int}(D')$, since B' homotopy retracts onto its 1-skeleton. The vector field X restricted to $s(B')$ projects under p_* to a nonsingular vector field Y on B'. The winding number of Y along $\partial B' = \partial D'$ is $\pm\chi(B) \neq 0$. Now the closed curve $s(\partial D')$ is homotopic in $p^{-1}(D')$ to a loop representing some element $\tilde{g}_0 \in \pi_1(L, x_0)$ which will have the same winding number $I_F(\tilde{g}_0) = \pm\chi(B) \neq 0$. □

Remark 4.1. The Hypothesis (0.2) is equivalent to the existence of the global section $\tilde{g}_1\colon M \to \Pi$ needed to define the global perturbed holonomy map $\tilde{H}_F(\tilde{g}_1)$. When there is a global section $s\colon B \to M$ of p, Hypothesis (0.2) can be replaced by the weaker assumption that there is a partial section $\tilde{g}_1\colon s(B) \to \Pi$ such that $\tilde{g}_1(s(y_0))$ generates $H_1(L; \mathbb{R})$.

4.2. Three examples showing the mutual independence of Theorems 4, 5 and 6. (None of these examples satisfies the hypotheses of the theorems of Langevin and Rosenberg [16] or Fuller [8]).

4.2.1. Let $M = S^3 \times (P^3 \# P^3)$, where $P \# P^3$ is the connected sum of two real projective spaces of dimension three. The fibration $h \circ p_1\colon S^3 \times (P^3 \# P^3) \to S^2$, where $h\colon S^3 \to S^2$ is the Hopf fibration and p_1 is the projection onto the first factor, has fiber $L = S^1 \times (P^3 \# P^3)$ null-homologous in M. The fundamental group of the fiber is $\pi = \pi_1(L, x_0) \approx \mathbb{Z} \times (\mathbb{Z}_2 * \mathbb{Z}_2)$ with commutator subgroup $[\pi, \pi] \approx \mathbb{Z}$. Theorem 4 shows that L is stable in the large, but neither Theorem 5 nor Theorem 6 applies.

4.2.2. Consider the Hopf fibration $h_2\colon S^5 \to \mathbb{C}P^2$ with fiber S^1. For some prime $p > 3$, pass to the quotient by the pth roots of unity to get $h_2\colon S^5/\mathbb{Z}_p \to \mathbb{C}P^2$ with fiber $S^1/\mathbb{Z}_p \approx S^1$. Let $M = S^5/\mathbb{Z}_p \times (P^3 \# P^3)$. The fiber $L = S^1/\mathbb{Z}_p \times (P^3 \# P^3)$ of the fibration $h_2 \circ p_1\colon M \to \mathbb{C}P^2$ is stable in the large by Theorem 5, but neither Theorem 4 nor Theorem 6 applies.

4.2.3. Let $M = S^5 \times P^3$ and consider the fibration $h_2 \circ p_1\colon S^5 \times P^3 \to \mathbb{C}P^2$. Then the fiber $S^1 \times P^3$ is stable in the large by Theorem 6, but neither Theorem 4 nor Theorem 5 applies.

4.3. Counterexamples when the hypothesis of Conjecture (0.1) are not satisfied.

4.3.1. Let $p\colon M_1 \to B$ be a smooth fibration with fiber S^1 given by the flow of a nonsingular vector field X on M_1 and suppose $\chi(B) = 0$. Then there exists a nonsingular vector field Y on

B, which we may suppose to have at most a finite number of periodic trajectories. Let Y be a vector field on M_1 which projects to Y under p_*. Then there exist arbitrarily small $\varepsilon>0$ such that the vector field $Z_\varepsilon=X+\varepsilon Y$ has no closed trajectories on M_1. If N is a smooth closed manifold with $H_1(N;\mathbb{R})=0$ then the foliation F_0 by the fibers of $p\circ p_1$: $M_1\times N\to B$ can be perturbed by multiplying the trajectories of Z by N to obtain a foliation F arbitrarily C^∞ close to F_0 without compact leaves. [A similar perturbation is possible for any fibration p: $M\to B$ with $\chi(B)=0$ provided that i^*: $H^1(M\colon\mathbb{R})\to H^1(L\colon\mathbb{R})\approx\mathbb{R}$ is surjective.]

4.3.2. In the previous example, if the condition $\chi(B)=0$ is replaced by the condition that B is noncompact, then a similar construction is possible using a vector field X on B with no periodic trajectories.

4.3.3. Let $L=S^1\times S^2\ \#\ S^1\times S^2$ and let $M=B\times L$, where B is any closed manifold. The product fibration p_1: $M\to B$ can be perturbed C^∞ to a foliation with no compact leaves, since $\pi_1(L,x_0)\approx\mathbb{Z}*\mathbb{Z}$ and the perturbed holonomy of the two generators can be chosen to be two arbitrary diffeomorphisms of B close to the identity with no common periodic point.

§5. GENERALIZATION OF THE KLEIN BOTTLE THEOREM

Proof of Theorem 7. Recall that M is the suspension of a diffeomorphism $f\colon L\to L$. This can be obtained from $\tilde{M}=L\times\mathbb{R}$ by the identification $(x,t)\sim(f(x),t-1)$ for all $(x,t)\in L\times\mathbb{R}$. Thus there is a covering map π: $\tilde{M}\to M$ and the projection p: $M\to S^1=\mathbb{R}/\mathbb{Z}$ is induced by the projection on the second factor p_2: $L\times\mathbb{R}\to\mathbb{R}$. We suppose that $f(x_0)=x_0$ for some point $x_0\in L$ (by modifying f by an isotopy, if necessary, so that the isotopy class of p: $M\to S^1$ is unchanged). Set $f_1=\pi\circ\tilde{f}_1$: $I\to M$ where $\tilde{f}_1$: $I\to\tilde{M}=L\times\mathbb{R}$ is defined $\tilde{f}_1(t)=(x_0,t), t\in I$. Let $\tilde{g}\in\pi_1(L,x_0)$ be an element whose real homology class $\bar{g}$ generates $H_1(L;\mathbb{R})$, and let $\tilde{g}_1\colon I\to\Pi$ be the lift of f_1 to the sheaf Π with $\tilde{g}_1(0)=\tilde{g}$. By Theorem 3′ with $K=I$ there exists a neighborhood V of the foliation F_0 by the fibers of p in $\mathrm{Fol}_q^1(M)$ such that for all $F\in V$ the perturbed holonomy map $H'_F(\tilde{g}_1)$: $I\to M$ is defined, and if for some $t\in I$ $H'_F(\tilde{g}_1)(t)=f_1(t)$, then the leaf L_{xF} of F through $x=f_1(t)$ is near L_x and diffeomorphic to L_x under π_x: $N_x\to L_x$. It remains to show that such a point $t\in I$ must exist for all $F\in V$ sufficiently close to F_0.

Suppose not, so that there is a sequence of foliations $F_n\in V$ converging to F_0 such that for all $t\in I$ and $n\geqslant 1$

$$H'_{F_n}(\tilde{g}_1)(t)\neq f_1(t). \tag{5.1}$$

Now $p\circ H'_{F_n}(\tilde{g}_1)$: $I\to\mathbb{R}/\mathbb{Z}$ can be lifted uniquely to a map h_n: $I\to\mathbb{R}$ such that $h_n(0)$ is near 0. In view of (5.1), h_n has no fixed points and we may suppose that $h_n(0)>0$ for all n (by passing to a subsequence, and replacing g by g^{-1}, if necessary).

We identify the fiber L with $p^{-1}(0)$, so that the basepoint x_0 of L is identified with $\pi(x_0,0)$, and we identify the transverse disk D_{x_0} at x_0 with a small interval around 0 in $\mathbb{R}$ by lifting p: $D_{x_0}\to\mathbb{R}/\mathbb{Z}$ to a diffeomorphism into $\mathbb{R}$. Then the perturbed holonomy at x_0 satisfies $H_{F_n}(\tilde{g})(0)=h_n(0)$.

Next, we repeat Thurston's construction of a homomorphism $\tilde{\varphi}$: $\pi_1(L,x_0)\to\mathbb{R}$, as in Lemma 2.4, with $x_n=0$ for all n. Let $g_1\in\Omega(L,x_0)$ be a loop representing the element $\tilde{g}\in\pi_1(L,x_0)$ chosen above. Choose loops $g_3,\ldots,g_r\in\Omega(L,x_0)$ whose real homology classes $\bar{g}_3,\ldots,\bar{g}_r\in H_1(L;\mathbb{R})$ vanish, such that the loops $g_1, g_2=g_1^{-1}, g_3,\ldots g_r$ form a symmetric set that generates $\pi_1(L,x_0)$. Let $d_n=\sup_i|H_{F_n}(g_i)(0)|$ and for each loop $g\in\Omega(L,x_0)$ that is not too long define $\varphi_n(g)=d_n^{-1}H_{F_n}(g)(0)$. Passing to a subsequence so that $\varphi_n(g_i)$ converges as $n\to\infty$ for each i, $1\leqslant i\leqslant r$, we find that $\varphi(g)=\lim_n\varphi_n(g)$ exists for all loops g and defines a nonzero homomorphism $\tilde{\varphi}$: $\pi_1(L, x_0) \to \mathbb{R}$ as in Lemma 2.4. Note that $\bar{\varphi}(\bar{g}) = \tilde{\varphi}(\tilde{g}) = 1$, where $\bar{\varphi}$: $H_1(L; \mathbb{R}) \to \mathbb{R}$ is the homomorphism induced by $\tilde{\varphi}$.

Since M is obtained from $L \times \mathbb{R}$ by the identification $(x, t) \sim (f(x), t-1)$, we see that $\tilde{g}_1(1) = f_* \tilde{g}_1(0) = f_*(\tilde{g})$, where f_* is the automorphism of $\pi_1(L, x_0)$ induced by f, so that the loop $f \circ g_1$ represents the element $\tilde{g}_1(1)$. Now $h_n(1) > 1$, which implies that $H_{F_n}(f \circ g_1)(0) > 0$ and consequently $\varphi_n(f \circ g_1) > 0$ for all n. In the limit $\bar{\varphi}(f_*(\bar{g})) = \tilde{\varphi}(f_*(\tilde{g})) \geqslant 0$. On the other hand, $f_*(\bar{g}) = c\bar{g}$ with $c < 0$ by hypothesis, so $\bar{\varphi}(f_*(\bar{g})) = c\bar{\varphi}(\bar{g}) < 0$, and the contradiction proves the theorem. □

In order to compare Theorem 7 with Plante's Theorem (Theorem 8), we recall the definition of the class $\mathscr{S}$ [19, 20]. Every group G which contains a finitely generated subgroup H with non-exponential growth [19], such that each $g \in G$ has a positive power $g^n \in H$, belongs to $\mathscr{S}$. Furthermore $\mathscr{S}$ is closed under extensions $1 \to G' \to G \to G'' \to 1$, and $\mathscr{S}$ is the smallest class with these properties.

Example 5.2. Let G be the "remarkable group" studied by Ghys and Sergiescu [9] and originally discovered by R. J. Thomson. The group G can be defined to consist of those homeomorphisms g of $S^1 = \mathbb{R}/\mathbb{Z}$ which are piecewise affine, whose left and right derivatives are integral powers of 2, and such that the point $g(0)$ and the points where the derivative of g is discontinuous are dyadic rationals (i.e. of the form $p/2^q$ mod $\mathbb{Z}$, where p and q are integers). The group is finitely presented, infinite, and simple. Furthermore it contains a subgroup that is a free group on two generators. [For example, take $f, g \in G$ such that each has exactly one fixed point, $f(1/4) = 1/4$ and $g(3/4) = 3/4$, with $f(1/2) = 1$ and $g(0) = 1/2$. Then f and g generate a free group, as is easily seen by considering the images of the intervals $(0, 1/2)$ and $(1/2, 1)$ mod $\mathbb{Z}$ under the action of distinct reduced words in f and g.] If a subgroup H of G contains positive powers f^m and g^n of f and g, then H contains the free group they generate and hence has exponential growth, so G cannot belong to $\mathscr{S}$.

Let N^4 be a smooth closed 4-manifold such that $\pi_1(N) \approx G$, let $L = S^1 \times N^4$ and let f: $L \to L$ be a diffeomorphism which reverses the orientation of S^1 and is the identity on N^4. Let p: $M^6 \to S^1$ be the suspension of f. Then Theorem 7 implies that the fiber L is stable in the large, since $H_1(L; \mathbb{R}) \approx \mathbb{R}$, but $\pi_1(L) \approx G \times \mathbb{Z}$ does not belong to $\mathscr{S}$, and so Theorem 8 is not applicable.

§6. ONE PARAMETER FAMILIES OF FOLIATIONS

N. Desolneux-Moulis [2, 3] has proven a form of stability in the large for perturbations within a smooth one parameter family of foliations starting with a fibration, under a certain initial transversality condition. We shall obtain a similar theorem with weaker hypotheses by using Theorem 3'.

Let $F(t)$ $(t \in \mathbb{R})$ be a C^2 one parameter family of foliations of a manifold M, i.e. they define a C^2 foliation of $M \times \mathbb{R}$ with leaves $L' \times \{t\}$ where L' is a leaf of $F(t)$. Letting $v \subset TM$ be the normal bundle to $F(0)$ orthogonal to its leaves, we define $\omega(t) \in \Omega^1(TF(0), v)$ to be the unique 1-form on $TF(0)$ with values in v, such that the graph of the homomorphism $\omega(t)_x$: $TF(0)_x \to v_x$ is the tangent plane $TF(t)_x$ for each $x \in M$ and $t \in \mathbb{R}$. There is a 1-form $\omega_1 \in \Omega^1(TF(0), v)$ defined by differentiating $\omega(t)$, $\omega_{1x} = \mathrm{d}/\mathrm{d}t\, \omega(t)_x|_{t=0}$.

Now suppose that $F(0)$ is the foliation F_0 by the fibers of a fibration p: $M \to B$. For each loop $g \in \Omega(L_x, x)$ we define the vector $X_g(\omega_1, x) \in TB_y$, where $x \in M$ and $y = p(x) \in B$, to be

$$X_g(\omega_1, x) = \int_0^1 p_* \omega_1(g'(s)) \mathrm{d}s$$

where we abbreviate $p_{*g(s)}$ and $\omega_{1g(s)}$ by p_* and ω_1, and $g'(s)$ is the velocity vector of g (assumed piecewise smooth) at $g(s)$. As in Proposition 6 of [3], we obtain

PROPOSITION 6.1. $X_g(\omega_1, x) = p_* \mathrm{d}/\mathrm{d}t\, H_{F(t)}(g)(x)|_{t=0}$.

Proof. The proof is a calculation in the product neighborhood $N_x \approx L_x \times D_y$, where we identify D_y with a disk centered at 0 in $\mathbb{R}^q$. Let g_t be the lift of g along π_x: $N_x \to L_x$ to the leaf of $F(t)$ through x, so that $g_t(0) = x$ and $g_t(1) = \mathrm{H}_{F(t)}(g)(x)$. Since $TF(t)_x$ is the graph of $\omega(t)_x$ we have

$$\begin{aligned} g_t'(s) &= g'(s) + \omega(t)_{g_t(s)}(g'(s)) \\ &= g'(s) + t\omega_1(g'(s)) + o(t). \end{aligned}$$

Applying p_* and differentiating with respect to t, we obtain

$$\mathrm{d}/\mathrm{d}t\, p_* g_t'(s)|_{t=0} = p_* \omega_1(g'(s)).$$

Integration of the LHS yields

$$p_* \mathrm{d}/\mathrm{d}t \int_0^1 g_t(s)\mathrm{d}s|_{t=0} = p_* \mathrm{d}/\mathrm{d}t\, H_{F(t)}(g)(x)|_{t=0}$$

since $\int_0^1 g_t(s)\mathrm{d}s = H_{F(t)}(g)(x) - x$, and integration of the RHS yields $X_g(\omega_1, x)$, by definition. □

Remark 6.2. For any compact leaf with trivial linear holonomy, the linear holonomy provides a canonical trivialization of the normal bundle DH: $v_L \to TD_{x_0}$. Then we may define $x_g(\omega_1, x_0) \in TD_{x_0}$ using DH in place of p_*, and Proposition 6.1 holds as before, with essentially the same proof.

COROLLARY 6.3. *If ω_1 and $y = p(x)$ are given, then $X_g(\omega_1, x)$ depends only on the real homology class $\bar{g} \in H_1(L_x; \mathbb{R})$ of g, and defines a homomorphism from $H_1(L_x; \mathbb{R})$ to TB_y.*

Proof. The perturbed holonomy $H_{F(t)}(g)(x)$ depends only on the homotopy class of g. Thus $X_g(\omega_1, x)$ defines a homomorphism from $\pi_1(L_x, x)$ to TB_y which is a real vector space, and the homomorphism factors uniquely through $H_1(L_x; \mathbb{R})$. □

As a consequence of the corollary, a section $\tilde{g}$: $M \to \Pi$ determines a vector field $X_{\tilde{g}}(\omega_1)$ on B, where $X_{\tilde{g}}(\omega_1)(y) = X_g(\omega_1, x)$, for any $x \in p^{-1}(y)$. Recall that the singularities of a vector field are generic if it is transverse to the zero section of the tangent bundle. For $X_{\tilde{g}}(\omega_1)$ this means intuitively that the family $F(t)$ changes with a nonzero initial velocity at time $t = 0$ along the loop g.

THEOREM 9. *In the fibration p: $M \to B$ suppose that $\chi(B) \neq 0$ and that there are a section $\tilde{g}$: $M \to \Pi$ and elements $\tilde{g}_2, \ldots, \tilde{g}_k \in \pi_1(L, x_0)$ such that $\tilde{g}_1 = \tilde{g}(x_0)$ is accessible in $\langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle$ and $\bar{g}_1, \ldots, \bar{g}_k$ generate $H_1(L; \mathbb{R})$. If $F(t)$ $(t \in \mathbb{R})$ is a C^2 one parameter family of foliations of M such that $F(0) = F_0$ and the vector field $X_{\tilde{g}}(\omega_1)$ is transverse to the zero section of the tangent bundle of B, then for all t sufficiently near 0 the foliation $F(t)$ has a compact leaf near to, and diffeomorphic to, some fiber.*

In fact, as the proof shows, there is a compact leaf near $p^{-1}(y)$ for every point $y \in B$ at which $X_{\tilde{g}}(\omega_1)$ vanishes, so there are at least $|\chi(B)|$ compact leaves.

In order to obtain the same conclusion, Desolneux–Moulis [3] supposes that $\pi_1(L_x, x)$ is abelian and that the sheaf Π is globally trivial (i.e. there are enough global sections of Π to generate $\pi_1(L, x_0)$), but she assumes transversality of $X_{\tilde{g}}(\omega_1)$ only for *some* section $\tilde{g}$ of Π. Theorem 9 remains valid under this weaker hypothesis provided we suppose that the subgroup $\langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle$ is abelian.

Proof of Theorem 9. Let K be the disjoint union of a finite number of compact disks B_a in B such that their interiors cover B. Choose $f: K \to M$ so that $p \circ f: K \to B$ restricts on each disk B_a to the inclusion into B, and so that $x_0 = f(z_0)$ for some $z_0 \in K$. Corresponding to the given section $\tilde{g}: M \to \Pi$ there is a lift $\tilde{g}'_1 = \tilde{g} \circ f: K \to \Pi$ of f. By translating $\pi_1(L, x_0)$ along paths in M to the disks $f(B_a)$ we obtain lifts $\tilde{g}'_i: K \to \Pi$ of f satisfying $\tilde{g}'_i(z_0) = \tilde{g}_i \in \pi_1(L, x_0)$, $i = 2, \ldots, k$. Now the hypotheses of Theorem 3′ are satisfied and for t sufficiently near 0 it suffices to find $z \in K$ such that $H'_{F(t)}(\tilde{g}'_i)(z) = f(z)$ to complete the proof.

Let $y_1, \ldots, y_d \in B$ be the points at which the vector field $X_{\tilde{g}}(\omega_1)$ vanishes, and choose z_j in the interior of K so that $pf(z_j) = y_j$. Since $\chi(B) \neq 0$, $X_{\tilde{g}}(\omega_1)$ must vanish, and the transversality hypothesis guarantees that $X_{\tilde{g}}(\omega_1)$ is hyperbolic, therefore with index ± 1, at each y_j. For t sufficiently small, we can conclude from Proposition 6.1 that there will be a unique hyperbolic fixed point x_{jt} of $H_{F(t)}(\tilde{g}_1(x_j))$ in a neighborhood U_j of $x_j = f(z_j)$ in D_{x_j}. For simplicity of notation suppose that $z_j = z_0$, so that $\tilde{g}_i(z_j) = \tilde{g}_i \in \pi_1(L, x_0)$ and $D_{x_j} = D$.

Next let $\langle \tilde{g}_1 \rangle = G_0 \lhd G_1 \lhd \ldots \lhd G_r \subseteq G = \langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle$ be a finite sequence of subgroups of $\pi_1(L, x_0)$ showing that $\tilde{g}_1$ is accessible in G. Replacing each $\tilde{g}_i$ by a power of itself, if necessary, we may suppose that $G_r = G$. Furthermore, possibly expanding the set of generators of G, we may assume that for every $\varepsilon = \pm 1$, $1 \le i \le k$ and $1 \le s < r$, the following statement holds:

$$\text{If } \tilde{g}_i \in G_{s+1}, \quad \text{then} \quad \tilde{g}_i^{\varepsilon} \tilde{g}_1 \tilde{g}_i^{-\varepsilon} \in \langle \tilde{g}_j : \tilde{g}_j \in G_s \rangle. \tag{6.4}$$

In fact, if (6.4) is not always satisfied, then we may add a new generator $\tilde{g}_i^{\varepsilon} \tilde{g}_1 \tilde{g}_i^{-\varepsilon}$ for every triple (ε, i, s) for which (6.4) fails. In case (6.4) fails for some of the new generators $\tilde{g}_i$, we repeat the process of adding new generators. Since the largest value of s for which there exist i and ε not satisfying (6.4) is decreased at each step, the process will terminate after a finite number of steps.

Now take t sufficiently small so that $H_{F(t)}(\tilde{g}_i)(x_{jt})$ is in U_j, $i = 1, \ldots, m$. By induction we show that for every $\tilde{g}_i \in G_s$, $H_{F(t)}(\tilde{g}_i)$ fixes x_{jt}. This is obvious for G_0, so suppose we have shown it for G_s, and let $g_i \in G_{s+1}$ be a generator. Now $\tilde{g}_i \tilde{g}_1 = \tilde{g}' \tilde{g}_i$ for some $\tilde{g}' \in G_s$ so

$$H_{F(t)}(\tilde{g}_1) H_{F(t)}(\tilde{g}_i)(x_{jt}) = H_{F(t)}(\tilde{g}_i)(x_{jt}) \in U_j,$$

for $\tilde{g}'$ is a monomial in the generators $\tilde{g}_i$ belonging to G_s and therefore $H_{F(t)}(\tilde{g}')$ fixes x_{jt}. Since x_{jt} is the only fixed point of $H_{F(t)}(\tilde{g}_1)$ in U_j, $H_{F(t)}(\tilde{g}_i)(x_{jt})$ must be x_{jt}, completing the inductive step. Thus x_{jt} is a common fixed point of the maps $H_{F(t)}(\tilde{g}_i)$. For t sufficiently small there will be a point z_{jt} near z_j on K, such that $f(z_{jt})$ and x_{jt} are joined by a short path on a leaf of $F(t)$. It follows that $H'_{F(t)}(\tilde{g}'_i)(z_{jt}) = f(z_{jt})$ for $i = 1, \ldots, k$, so Theorem 3′ yields the desired conclusion.

Note that if $G = \langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle$ is commutative and $X_{\tilde{g}}(\omega_1)$ is generic for some $\tilde{g} \in G$, then $\tilde{g}$ may be used instead of $\tilde{g}_1$, and the proof proceeds just as before. □

Remark. In view of Corollary 6.3, the hypothesis of Theorem 9 affirming that $\tilde{g}_1$ extends to a section $\tilde{g}: M \to \Pi$ can be weakened to the assumption that $\bar{g}_1 \in H_1(L; \mathbb{R})$ extends to a section of the locally trivial sheaf of first real homology groups of the fibers over B. The proof proceeds just as before, merely replacing Π by the sheaf of homology groups.

§7. LOCALLY STABLE COMPACT LEAVES

In this section we prove Theorems 11 and 12 concerning local stability of compact leaves and give some examples.

Proof of Theorem 11. Apply Theorem 1. Then it will suffice to show that for all F in a sufficiently small neighborhood $V_1 \subseteq V$ of F_0, there is a common fixed point $x \in U$ of the perturbed holonomy maps $H_F(g_i)$, $i = 1, \ldots, k$.

Accessibility of $\tilde{g}_1$ in $G = \langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle$ means that there is a finite normal series $G_0 \lhd G_1 \lhd \ldots \lhd G_r$ where $\tilde{g}_1$ generates G_0 and as in the proof of Theorem 9, we may assume that $G_r = G$ and that (6.4) is always satisfied. Extend the set $\{\tilde{g}_1, \ldots, \tilde{g}_k\}$ to a finite symmetric subset of G, $\{\tilde{g}_1, \ldots, \tilde{g}_m\}$, by adding inverses whenever necessary. Let $D' \subset U$ be a closed interval neighborhood of x_0. Choose V_1 small enough so that for every $F \in V_1$, the following conditions hold.

7.1 $H_F(\tilde{g}_1)$ has a fixed point in D' and no fixed point in $U - D'$.

7.2. If we set $h_i = H_F(\tilde{g}_i)$, the perturbed holonomy $h_i : U \to D$ is defined and $h_i(D') \subseteq U$ for $i = 1, \ldots, m$.

Consequently, identifying D with an interval in $\mathbb{R}$ we find that the set A of fixed points of h_1 is a nonempty closed subset of D' such that

$$\{a, b\} \subseteq A \subseteq [a, b] \subseteq D'$$

where $a = \inf(A)$ and $b = \sup(A)$.

We prove inductively that for every $\tilde{g} \in G_s$, $h = H_F(\tilde{g})$ fixes a and b. This is clear for G_0. Suppose it holds for G_s and let $\tilde{g}_i \in G_{s+1}$. Then $\tilde{g} = \tilde{g}_i \tilde{g}_1 \tilde{g}_i^{-1}$ belongs to G_s and its perturbed holonomy fixes a, so $h_i(a) = h_1(h_i(a))$. The fixed point $h_i(a)$ of h_1 belongs to U and consequently to A. Arguing similarly for b we conclude that $h_i([a, b]) \subseteq [a, b]$. Since g_i is also a generator of G_{s+1} we find similarly that $h_i^{-1}\,[a, b] \subseteq [a, b]$, so that h_i fixes a and b. Thus G_{s+1} fixes a and b, completing the induction. Hence a and b are common fixed points of $H_F\,(\tilde{g}_i)$, $i = 1, \ldots, k$. □

Now suppose that $F(t)$ is a C^2 one parameter family of codimension q foliations on M, and let L_0 be a compact leaf of $F_0 = F(0)$ with trivial linear holonomy. For an element $\tilde{g} \in \pi_1(L_0, x_0)$ define $A(\tilde{g}) \in \mathrm{End}\,(TD_{x_0})$ by

$$A(\tilde{g}) = \mathrm{d}/\mathrm{d}t\,\{DH_{F(t)}(\tilde{g})_{x_0}\}|_{t=0},$$

and consider the following hypothesis.

HYPOTHESIS 7.3. *$A(\tilde{g})$ has no eigenvalues on the imaginary axis $i\mathbb{R}$.*

Intuitively (7.3) means that the initial velocity of change of the perturbed holonomy is such that $H_{F(t)}(\tilde{g})$ is hyperbolic for all nonzero t near 0. Now in the spirit of Desolneux-Moulis' Theorem [3] and our Theorem 9 we have the following.

THEOREM 12. *With $F(t)$ and L_0 as above, suppose that there are elements $\tilde{g}_1, \ldots, \tilde{g}_k \in \pi_1(L_0, x_0)$ such that $\tilde{g}_1, \ldots, \tilde{g}_k$ generate $H_1(L_0; \mathbb{R})$, $\tilde{g}_1$ is accessible in $\langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle$, and Hypothesis (7.3) holds with $\tilde{g} = \tilde{g}_1$. Then for all t sufficiently close to 0, $F(t)$ has a compact leaf near to and diffeomorphic to L_0.*

Proof. Denote $DH_{F(t)}(\tilde{g}_1)_{x_0}$ by $A(t)$. Since $A(0)$ is the identity I we may write $A(t) = \exp B(t)$, where $B(t)$ is a C^1 curve in $\mathrm{End}(TD_{x_0})$ with $B(0) = 0$. Now $B'(0) = A'(0) = A(\tilde{g})$ so the eigenvalues of

$$B(t) = tB'(0) + o(t), \quad t \neq 0,$$

have nonvanishing real part. Consequently $A(t)$ has no eigenvalues on S^1, so $A(t)$ is hyperbolic and $H_{F(t)}(\tilde{g}_1)$ has a unique fixed point $x_t \in D$ for t near 0. Since $\tilde{g}_1$ is accessible in $\langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle$ it follows (as in Theorems 9 and 11) that x_t is also a fixed point of $H_{F(t)}(\tilde{g}_i)$ for $i=2, \ldots, k$, and the Theorem follows from Theorem 1. □

Example. 7.5. If H is an infinite, simple, nonabelian finitely presented group and $G=\mathbb{Z}*H$, then G has no accessible elements. Let L_0 be a compact smooth manifold with $\pi_1(L_0, x_0)=G$ (always possible if $\dim(L_0)\geqslant 4$). Cut L_0 along a non-separating closed codimension one submanifold N to get L' such that $\partial L'$ has two components $\partial_0 L'$ and $\partial_1 L'$. Let h be a weakly hyperbolic diffeomorphism of $(\mathbb{R}^q, 0)$ and let M be the quotient of $L' \times \mathbb{R}^q$ by $(x, y) \sim (x', h(y))$ where $x \in \partial_0 L'$ and x' is the corresponding point in $\partial_1 L'$, with the foliation F_0 induced by the product foliation $L' \times \{\text{point}\}$. Then L_0 is a stable compact leaf by Theorem 10 but no theorems of [13] are applicable.

Example 7.6. Let T^n be the n-dimensional torus and multiply the previous example by T^n. If $q=1$ then by Theorem 11 $L_0 \times T^n$ is a locally stable compact leaf of the foliation $F_0 \times T^n$ on $M \times T^n$, but the results of [13] are not applicable.

Example 7.7. The leaf $L_0 = S^1 \times S^2 \# S^1 \times S^2$ of (4.3.3) can never be a stable compact leaf because the two generators of $\pi_1(L_0, x_0)=\mathbb{Z}*\mathbb{Z}$ can have entirely independent holonomy. Thus some hypothesis on the fundamental group is necessary to have local stability.

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APPENDIX

In this Appendix we prove a number of lemmas related to perturbed holonomy including Lemmas 8.1, 1.1 and 2.1 and Propositions 1.4 and 3.1, in that order. All paths and loops are assumed to be piecewise smooth.

LEMMA 8.1. *Let L be a compact Riemannian manifold and let $g_1, \ldots, g_k \in \Omega(L, x_0)$ be loops which generate $\pi_1(L, x_0)$. Given $C > 0$ there exists $C' > 0$ such that every loop $g \in \Omega(L, x_0)$ of length less then C is homotopic to a product of generators (and their inverses) such that each intervening loop in the homotopy has length less than C'.*

Proof. Choose a smooth triangulation of L such that x_0 is a vertex and the open star $V(x)$ of each vertex x is geodesically convex with diameter less than 1. Take $n > C/\delta$ where δ is the Lebesgue number of the star cover. We claim

8.2. Every loop $g \in \Omega(L, x_0)$ of length less than C is homotopic to a simplicial loop through loops of length less than $4n$.

Proof. Write $g = g_1 \ldots g_n$ where each g_i is a path in some star, say $V(x_i)$. Between g_i and g_{i+1} insert the null-homotopic loop $a_i h_i b_{i+1}$ where a_i is the geodesic in $V(x_i)$ from $g_i(1)$ to x_i, h_i is the 1-simplex from x_i to x_{i+1}, and b_{i+1} is the geodesic from x_{i+1} to $g_{i+1}(0)$. Then

$$g = g_1 \ldots g_n \simeq (\Pi_i g_i a_i h_i b_{i+1}) g_n \simeq h_1 \ldots h_n$$

where the product is taken over $1 \leqslant i \leqslant n-1$. The null-homotopies of $a_i h_i b_{i+1}$ and $b_i g_i a_i$ can be chosen to have intermediate loops of length less than 3. □

The lemma follows, since the number of simplicial loops of length less than $4n$ is finite, and we can choose an arbitrary homotopy to a product of generators and inverses for each. □

By adapting (8.2) to paths we obtain

LEMMA 8.3. *Given a compact Riemannian manifold L and $C>0$ there exists $C'>0$ such that two homotopic paths g_0, g_1: $I\to L$ of length less than C are homotopic such that the intervening paths have length less than C'.*

In order to establish Lemma 1.1 we shall prove the slightly stronger statement which follows. As in §1, L_0 is a compact leaf of the C^1 codimension q foliation F_0 on M, and we have fixed a Riemannian metric on M and constructed the tubular neighborhood π: $N\to L_0$ of radius $\varepsilon>0$. For $0<\delta\leqslant\varepsilon$ let $N(\delta)$ be the tubular subneighborhood of L_0 of radius δ and let $D_y(\delta)=D_y\cap N(\delta)$ where $D_y=\pi^{-1}(y)$, $y\in L_0$.

LEMMA 8.4. *Given $C'>0$ there exist a neighborhood V of F_0 in $Fol_q^1(M)$ and $\delta>0$ with the following property. For every $F\in V$ and $x\in N(\delta)$, every path g in L_0 of length less than C' starting at $\pi(x)$ lifts along π to a path g' starting at x and lying on a leaf of F.*

Proof. We shall prove (8.4) by using the basic neighborhoods of Epstein [7], rather than the axioms he gives, since Axiom 2 of [7] considers only a single path.

As in the proof of Lemma 8.1 we choose a triangulation of L_0 with the properties indicated there. Let the vertices be $x_1,\ldots x_r$. As in [7] choose a *neighborhood scheme* $S=(I,\{\varphi_i\},\{K_i\})$ for F_0; i.e. $\varphi_i: U_i\to\mathbb{R}^n$ for $i\in I$ is a C^1 diffeomorphism from an open set U_i in M to an open set in $\mathbb{R}^n$, so that for each leaf L of F_0 each component of $\varphi_i(L)$ is parallel to $\mathbb{R}^{n-q}\times\{0\}$; $n=\dim(M)$ and $K_i\subset U_i$ is such that $\varphi_i(K_i)$ is a closed n-cube with sides parallel to the axes in $\mathbb{R}^n$, the family $\{K_i\}$ is locally finite, and $\{\mathrm{Int}(K_i)\}$ covers M. Call any pair $(\varphi: U\to\mathbb{R}^n, K)$ with the above properties a *distinguished chart* on M for F_0. Make the choice so that $I_0=\{1,\ldots,r\}\subseteq I$, so that these are precisely the indices i for which U_i meets N, and so that for $i\in I_0$, $U_i\cap L$ is the open star of the vertex x_i and K_i contains $\pi^{-1}(K_i\cap L)$. Without loss of generality we may assume that *every* distinguished chart (φ, K) has $\varphi(K)=I^n$, the unit cube in $\mathbb{R}^n$.

Given S, $i\in I$ and $\varepsilon'>0$, we define a neighborhood $N(i,\varepsilon')$ of F_0 in $\mathrm{Fol}_q^1(M)$ to consist of those foliations F such that there is a distinguished chart (φ', K) for F with $K_i\subseteq K$ and

$$|\varphi'\circ\varphi_i^{-1}-Id|_1<\varepsilon' \quad \text{on } \varphi_i(K_i)=I^n \tag{8.5}$$

where $|\bullet|_1$ denotes the C^1 norm, defined to be the supremum of the absolute values of the function and its first partial derivatives. A basis of the compact C^1 topology of [7] is given by finite intersections of such neighborhoods $N(i,\varepsilon')$ for S. [Epstein does not require $\varphi(K)=I^n$, but by taking a smaller ε' we may make that assumption, at least for the C^1 topology.]

Case 1. We consider only paths g which lie in K_i for a fixed $i\in I_0$. Carry all the structures over into the open neighborhood $\varphi_i(U_i)$ of I^n by φ_i. For example, $\pi^{-1}(K_i)\subset N$ passes over to a tubular ε-neighborhood of $\varphi_i(K_i\cap L)=I^{n-q}\times\{0\}$, relative to the induced metric (not necessarily Euclidean), and $\psi=\varphi'\circ\varphi_i^{-1}: I^n\to I^n$ will be a diffeomorphism onto a subset of I^n with $|\psi-Id|_1<\varepsilon'$ for some ε' to be chosen. Given ε_1, $0<\varepsilon_1\leqslant\varepsilon$, choose δ_1 such that $I^{n-q}\times[-\delta_1,\delta_1]$ is contained in the ε_1-neighborhood of $I^{n-q}\times\{0\}$, choose $\varepsilon'>0$ such that all ψ: $I^n\to I^n$ with $|\psi-Id|_1<\varepsilon'$ satisfy

$$\psi(I^{n-q}\times[-\delta_1/2,\delta_1/2])\subseteq I^{n-q}\times[-\delta_1,\delta_1],$$

and choose $\varepsilon_2>0$ such that the ε_2-neighborhood of $I^{n-q}\times\{0\}$ is contained in $I^{n-q}\times[-\delta_1/2, \delta_1/2]$. Then it is clear that for every $F\in N(i,\varepsilon')$ and every $x\in K_i\cap N(\varepsilon_2)$, any path g in $K_i\cap L$ with $g(0)=\pi(x)$ lifts to a path g' starting at x and lying on a leaf of F, completing Case 1. Note also that

8.6. The perturbed holonomy $H_F(g)$ of all paths g in $L\cap K_i$ tends uniformly to the identity in the C^1 norm, as ε' tends to 0.

This is clear from (8.5).

It follows easily, by intersecting the neighborhoods $N(i,\varepsilon')$ for $i=1,\ldots,r$ (where ε' depends on i) and letting $\alpha(\varepsilon_1)$ be the least of the corresponding numbers ε_2, that we can find a neighborhood V_0 of F_0 such that

8.7. Every path g contained in some $K_i\cap L$ lifts to every x in $\pi^{-1}(g(0))\cap N(\alpha(\varepsilon_1))$ for every $F\in V_0$.

The general case. Fix $n > C'/\delta$ (where δ is the Lebesgue number of the cover $\{\mathrm{Int}(K_i) \cap L\}$ of L, as in the proof of (8.1). Let $\delta_j = \alpha^{n-j}(\varepsilon), j = 0, \ldots, n$ and by repeatedly applying Case 1 choose neighborhoods $V_n \supseteq V_{n-1} \supseteq \ldots \supseteq V_1$ of F_0 so that (8.7) holds with V_0 replaced by V_j and $\varepsilon_1 = \delta_j, j = 1, \ldots, n$. Since any path in L of length less than C' can be decomposed $g = g_1 \ldots g_n$ where each g_j lies in some $K_i \cap L$, repeated application of (8.7) shows that the lemma holds for $V = V_1$ and $\delta = \delta_0$. □

Proof of the Compact Leaf Stability Lemma 2.1. If we knew that the leaf L of F containing x were contained in the tubular neighborhood N, the proof would be an easy argument in covering space theory using $\pi: N \to L_0$ restricted to L. This proof does in fact work, as we now show, provided that we keep track of the lengths of the paths involved so that they cannot leave N.

For a fixed Riemannian metric on M, choose

$$C > 2 \cdot \text{diameter } (L_0) + \max_{1 \leq j \leq m} \text{length } (g_j),$$

let $C' \geq C$ be the associated constant satisfying Lemma 8.1, and apply Lemma 1.1 to get neighborhoods V of F_0 in $\mathrm{Fol}_q^1(M)$ and U of x_0 in D such that for any $F \in V$ and $x \in U$, every path g in L_0 of length less than C' starting at x_0 lifts along π to a path g' starting at x and on the leaf L of F passing through x. Clearly (1′) is satisfied by $V_1 = V$ and $U_1 = U$.

Now suppose that for some $F \in V$ and $x \in U$ we have $H_F(g_j)(x) = x$ for each j. Define a diffeomorphism s: $L_0 \to L$ by setting

$$s(y) = H_F(g)(x), \quad y \in L_0$$

where g is a path of length less than $C/2$ from x_0 to y on L_0. If we show that s is well-defined then (since F is transverse to the fibers of $\pi: N \to L_0$), s will be a smooth bijection with smooth inverse $\pi|_L$. The image $s(L_0)$ will be all of L since it is both open and closed in L.

To show that s is well-defined, let g_1 be another path of length less than $C/2$ from x_0 to y. The loop $g_1 g^{-1}$ is homotopic to a product of generators g_j so that each intermediate loop lifts, and then $H_F(g_1 g^{-1})$ fixes x since the perturbed holonomy $H_F(g_j)$ of each generator does. Thus $H_F(g_1)(x) = H_F(g)(x)$ and s is well-defined. □

Proofs of Propositions 1.4 *and* 3.1. Proposition 1.4 is the special case of Proposition 3.1 in which $K = \{x_0\}$ and $\tilde{g}(x_0)$ is the homotopy class of g, so it suffices to prove 3.1. As in the proof of 8.4 we proceed by cases, and we use the constructions and notation of that proof.

Case 1. Suppose that there is a map g: $K \times I \to \mathrm{Int}(K_i)$ for some i such that the path g_z defined $g_z(t) = g(z, t)$ represents $\tilde{g}(z)$ for each $z \in K$. Then $H_F(\tilde{g})$ is well-defined and continuous by Lemma 8.4. If $\tilde{g}$ is C^1 we may choose g to be C^1 and then $H_F(\tilde{g})$ is clearly C^1. Continuity of H follows from the observation (8.6) made during the proof of Lemma 8.4.

Case 2. Suppose that there are maps g_j: $K \times I \to \mathrm{Int}(K_{i(j)}) \cap L$, $j = 1, \ldots, n$, such that $g_j(z, 1) = g_{j+1}(z, 0)$ for all z and j, and such that the path $g_z = g_{1z} \ldots g_{nz}$ (where $g_{jz}(t) = g_j(z, t)$) represents $\tilde{g}(z)$ for all $z \in K$. This case follows by repeated application of Case 1.

The general case. In some neighborhood K' of $z' \in K$ representative paths for $\tilde{g}$ can be chosen by a continuous map g: $K' \times I \to L$. Decompose $g_{z'} = g_{1z'} \ldots g_{nz'}$ for some n so that each path g_{jz} lies in $\mathrm{Int}(K_{i(j)}) \cap L$ for some $i(j)$. Possibly reducing the size of the neighborhood K', we may extend each $g_{jz'}$ to a map g_j: $K' \times I \to \mathrm{Int}(K_{i(j)}) \cap L$ and apply Case 2 to obtain the desired conclusions for $\tilde{g}|_{K'}$. By the compactness of K a finite number of such sets K' cover K and we may let V be the intersection of the corresponding neighborhoods. □